**THE CHINESE UNIVERSITY OF HONG KONG**
**MATH3270B**
**HOMEWORK2 SOLUTION**

**Question 1**:

(1) The characteristic equation is $r^2 + 8r - 9 = 0$ i.e. $(r+9)(r-1) = 0$, which has two roots $r_1 = -9$, $r_2 = 1$. Hence, the general solution is $y = C_1 e^{-9t} + C_2 e^{t}$, where $C_1$ and $C_2$ are constants.

(2) The characteristic equation is $r^2 + 4r + 4 = 0$ i.e. $(r+2)^2 = 0$, which has two same roots $r_1 = r_2 = -2$. Hence, the general solution is $y = (C_1 + C_2 t)e^{-2t}$, where $C_1$ and $C_2$ are constants.

(3) The characteristic equation is $r^2 + 5r + 8 = 0$, which has two imagine roots $r_1 = -\frac{5}{2} + \frac{\sqrt{7}}{2}i$, $r_2 = -\frac{5}{2} - \frac{\sqrt{7}}{2}i$. Hence, the general solution is $y = e^{-\frac{5}{2}t}(C_1 \sin(\frac{\sqrt{7}}{2}t) + C_2 \cos(\frac{\sqrt{7}}{2}t))$, where $C_1$ and $C_2$ are constants.

**Question 2**:

(1) For the second order O.D.E $y'' + p(t)y' + q(t)y = 0$, suppose the O.D.E has two linearly independent solutions $y_1$ and $y_2$, by the definition of Wronskian, we have $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$, we differentiate on the both sides and use the proposition that $y_1, y_2$ are the solutions of the O.D.E, we get

$$W'(t) = y_1 y_2'' - y_2 y_1'' = -p(t)W(t),$$

we solve this separable O.D.E(note that $W(t)$ is nonzero because the former O.D.E is second order), we get $W(t) = e^{-\int p(t)}$. Hence, if the Wronskian is constant, then we must have $p(t) = 0$.

(2) By direct calculation, we have

$$\begin{aligned} W(fg, fh) &= fg(fh)' - fh(fg)' \\ &= fg(f'h + fh') - fh(f'g + fg') \\ &= f^2(gh' - hg') \\ &= f^2 W(g,h). \end{aligned} \tag{1}$$

**Question 3**:

(1) Form 2.1, we know that the Wronskian satisfies an O.D.E, $W'(x) = p(x)W(x)$, and we also deduce the formula that $W(x) = e^{-\int p(x)}$. Hence, in the question, $p(x) = \frac{1}{x}$, we get $W(x) = C/x$, where C is a nonzero constant.

(2) Similarly, in the equation, we have $p(x) = \frac{2x}{x^2-1}$, hence we deduce that $W(x) = C/(x^2-1)$, where C is a nonzero constant.

**Question 4**:

(1) We consider the general case, if the second order O.D.E $a(t)y'' + b(t)y' + y = 0$ has a known sol $y_1$. We assume $y_2 = v(t)y_1$ is another independent sol, we differentiate $y_2$ twice and substitute into the equation, we deduce that

$$a(t)[v''y_1 + v'y_1'] + b(t)v'y_1 + v(a(t)y_1'' + b(t)y_1' + y_1) = 0,$$

now since $y_1$ is a sol, we deduce finally that $v$ satisfies the O.D.E:

$$a(t)[v''y_1 + v'y_1'] + b(t)v'y_1 = 0.$$

In question (1), we have

$$t^2[v''t^{-1} - 2v't^{-2}] + 3tv't^{-1} = 0,$$

i.e. $tv'' + v' = 0$. Let $w = v'$ and $w$ satisfies $tw' + w = 0$. We solve this equation and deduce $w = \frac{c}{t}$, where c is a constant. Hence, $v = c\ln t + k$, where c and k are constants. Finally, $y_2 = ct^{-1}\ln t + kt^{-1}$, since the second part is a multiple of $y_1$ and can be dropped, but the first provides a new sol $y_2 = t^{-1}\ln t$. One can check that the Wronskian of $y_1$ and $y_2$ is $W(y_1, y_2) = t^{-3} > 0$. So $y_2 = t^{-1}\ln t$ is a new independent sol.

(2) Form 4.1, we have in this equation, $v$ satisfies the O.D.E:

$$(x-1)[v''e^x + 2v'e^x] - xv'e^x = 0,$$

i.e. $(x-1)v'' + (x-2)v' = 0$. We solve this differential equation and deduce that $v(x) = cxe^{-x} + k$, where c and k are constants. Finally, $y_2 = cx + ke^x$, we drop the second term and choose $y_2 = x$. One can check that the Wroskian of $y_1$ and $y_2$ is $W(y_1, y_2) = (x-1)e^x > 0$.

**Question 5**:

(1) We have already know:
the sol of inhomogeneous equation=the general sol of homogeneous equation+ a particular sol of the inhomogeneous equation.
In this question, the corresponding homogeneous equation is $y'' + 3y' + 2y = 0$, we deduce

the general sol of homogeneous equation is $y = C_1e^{-t} + C_2e^{-2t}$. Now we separate the inhomogeneous equation into three equations:

$$y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t, \tag{2}$$

$$y'' + 3y' + 2y = 3e^{-t}\cos t, \tag{3}$$

$$y'' + 3y' + 2y = 6e^t. \tag{4}$$

Now if we find the particular sol of (2), (3), (4), denoted to be $y_1, y_2, y_3$, then $y^* = y_1 + y_2 + y_3$ is a particular sol of the ordinary equation. Since $1 + 2i$, $-1 + i$, 1 are not the roots of the characteristic equation of ordinary equation, we assume

$$y^* = e^t[\sin 2t(a_2t^2 + a_1t + a_0) + \cos 2t(b_2t^2 + b_1t + b_0)] + e^{-t}(c_0\sin t + d_0\cos t) + f_0e^t,$$

we differentiate twice and substitute into the ordinary equation, finally to get $a_2 = \frac{1}{52}$, $a_1 = \frac{10}{169}$, $a_0 = -\frac{1233}{35152}$, $b_2 = -\frac{5}{52}$, $b_1 = \frac{73}{676}$, $b_0 = -\frac{4105}{35152}$, $c_0 = -\frac{3}{2}$, $d_0 = \frac{3}{2}$, $f_0 = \frac{2}{3}$. Finally, the sol of inhomogeneous equation is $y = C_1e^{-t} + C_2e^{-2t} + y^*$, where the $y^*$ is given above.

(2) Similarly, the general sol of corresponding homogeneous equation is $y = e^{-t}(C_1\sin 2t + C_2\cos 2t)$. Now since $-1 + 2i$ is the root of the characteristic equation, $-2 + i$ is not the root of the characteristic equation, we assume

$$y^* = te^{-t}[(a_1t + a_0)\sin 2t + (b_1t + b_0)\cos 2t)] + e^{-2t}[(c_1t + c_0)\sin t + (d_1t + d_0)\cos t],$$

we differentiate twice and substitute into the ordinary equation, we deduce that $a_1 = \frac{3}{8}$, $a_0 = 0$, $b_1 = 0$, $b_0 = \frac{3}{16}$, $c_1 = \frac{1}{5}$, $c_0 = \frac{1}{25}$, $d_1 = -\frac{2}{5}$, $d_0 = -\frac{7}{25}$. Finally, the sol of inhomogeneous equation is $y = y = e^{-t}(C_1\sin 2t + C_2\cos 2t) + y^*$, where the $y^*$ is given above.